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Variational inequalities with operator solutions ⋆

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Abstract. The aim of the present paper is to give a new kind of point of view in the theory of variational inequalities. Our approach makes possible the study of both scalar and vector variational inequalities under a great variety of assumptions. One can include here the variational inequalities defined on reflexive or nonreflexive Banach spaces, as well as the vector variational inequalities defined on topological vector spaces.

Key words: variational inequalities, pseudomonotonicity, KKM-mapping

1. Introduction

We start with the classical formulation of the variational inequalities.
Let X be a Banach space, X^* its dual space, $K \subset X$ be a nonempty, convex set, and $T : K \to X^*$ be a mapping. The variational inequality problem is:

$$\begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle T(x_0), x - x_0 \rangle \geqslant 0, \forall x \in K. \end{cases} \qquad (VI)$$

A very important factor in the solvability of (VI) is the weak-compactness of the closed, convex, bounded subsets of a reflexive Banach space. A possible treatment in nonreflexive Banach spaces has been given in [3], where taking into consideration the inclusion $K \subset X^{**}$, the mapping T was defined on a subset of X^{**}, and solutions in X^{**} were obtained. In this way the weak-compactness of bounded and weakly closed subets of X^{**} is guaranteed, but some cases are not covered, for example when $T : L^\infty(\Omega) \to L^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

As an improvement we recommend the setting $T : K \subset X^* \to X$. This will appear as a special case of our (OVI) problem below and covers any case mentioned above. The infinite dimensional formulation of vector variational inequalities [4, 9, 14, 19] uses Banach spaces Z, W, a nonempty, closed, convex subset $K \subset Z$ a mapping $T_1 : K \to (Z, W)^*$, where by $(Z, W)^*$ we denoted the space of all linear and continuous mappings from Z into W. Let $C_1 : K \rightsquigarrow W$ be a set-valued mapping with $C_1(z)$ a closed, convex, pointed cone such that $\text{int}C_1(z) \neq \emptyset$, for all $z \in K$. With $\text{int}C_1(z)$ we denote the interior in the norm topology.

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The vector variational inequality can be stated as:

$$\begin{cases} \text{Find } z_0 \in K \text{ such that} \\ \langle T_1(z_0), z - z_0\rangle \notin -\text{int}C_1(z_0), \forall z \in K. \end{cases} \qquad (VVI)$$

Problems of this kind were studied recently in [4, 9, 12, 13, 17, 19].
Let us recall a recent result concerning the existence of the solutions of (VVI).

DEFINITION 1.1. [19] The mapping $T_1 : K \to (Z, W)^*$ is said to be weakly C_1-pseudomonotone if for each $z, u \in K$

$$\langle T_1(z), u - z\rangle \notin -\text{int}C_1(z) \Rightarrow \langle T_1(u), u - z\rangle \notin -\text{int}C_1(z).$$

DEFINITION 1.2. The mapping $T_1 : K \to (Z, W)^*$ is said to be hemicontinuous on K if for every $z, u \in K$, $t \in [0, 1]$ the mapping

$$t \mapsto \langle T_1 (z + t(u - z)), u - z\rangle$$

is continuous at 0^+.

DEFINITION 1.3. [19] The mapping $T_1 : K \to (Z, W)^*$ is said to be coercive if there exists aweakly compact subset B of Z and $u_0 \in B \cap K$ such that

$$\langle T_1(z), u_0 - z\rangle \in -\text{int}C(z), \forall z \in K \setminus B.$$

THEOREM 1.1. [19] *Let Z, W be real Banach spaces. Let K be a nonempty closed, convex subset of Z. Let $C_1 : K \rightsquigarrow W$ be such that, for each $z \in K$, $C_1(z)$ is a closed, convex pointed cone with int$C_1(z) \neq \emptyset$, and let $W \setminus (-\text{int}C_1(\cdot))$ having a weakly closed graph in $Z \times W$. Suppose that $T_1 : K \subset Z \to (Z, W)^*$ is weakly C_1-pseudomomotone, hemicontinuous on K and coercive. Then the (VVI) has a solution.*

REMARK 1.1. Under the assumptions of Theorem 1.1, the Banach space W has to be finite dimensional. Indeed, the assumption that $W \setminus (-\text{int}C_1(\cdot))$ has a weakly closed graph, implies that $W \setminus (-\text{int}C_1(z))$ is weakly closed for each $z \in K$ and hence int$C_1(z)$ is weakly open for each $z \in K$. But $C_1(z)$, as a closed, convex, pointed cone, has interior points in the weak topology only if the space W is finite dimensional.

Our goal is to give a suitable approach of the above mentioned problems in a wider context. Let us show now our general setting. Let X, Y be Hausdorff topological vector spaces. By $(X, Y)^*$ we denote the space of linear and continuous operators from X into Y endowed with the topology of pointwise convergence.

A subset $\tilde{C}$ of Y is called a convex cone if $\tilde{C} + \tilde{C} \subset \tilde{C}$ and $\lambda\tilde{C} \subset \tilde{C}$ for all $\lambda > 0$. Let $K \subset (X, Y)^*$ be a nonempty, convex set, $T : K \subset (X, Y)^* \to X$ be a mapping, and $C : K \rightsquigarrow Y$ be a set-valued mapping with convex, cone values such that $0 \notin C(f)$for all$f \in K$.

The variational inequality problem which we intend to solve is:

$$\begin{cases} \text{Find } f_0 \in K \text{ such that} \\ \langle f - f_0, T(f_0) \rangle \notin C(f_0), \forall f \in K. \end{cases} \qquad (OVI)$$

The notation (OVI) is motivated by the fact that the solutions are sought in the space of linear and continuous operators.

In the next section we will study the solvability of the problem (OVI) using the C-psedomonotonicity assumption (Theorem 2.1) and we will show that, using our result, a stronger version of Theorem 1.1 can be proved (Corollary 2.1).

In section 3 we study, as a special case of (OVI), the solvability of scalar variational inequalities defined on Hausdorff topological vector spaces, and we solve a variational inequality defined on $L^{\infty}(\Omega)$.

In section 4 we introduce the notion of B-pseudomonotonicity, as a natural generalization of the pseudomonotonicity introduced by Brézis. We will prove the solvability of (OVI) using the B-pseudomonotonicity (Theorem 4.1) and a connection between the C and B-pseudomonotonicities. These results are significant even in the setting of the problems (VI) and (VVI).

2. Solvability of OVI

Let us consider our setting for the (OVI) variational inequalities from the previous section.

DEFINITION 2.1. The mapping $T : K \to X$ is said to be C-pseudomonotone, if

$$\langle g - f, T(f) \rangle \notin C(f) \Rightarrow \langle g - f, T(g) \rangle \notin C(f).$$

DEFINITION 2.2. The mapping $T : K \to X$ is said to be hemicontinuous, if the function

$$t \mapsto \langle g - f, T(f + t(g - f)) \rangle$$

is continuous at 0^+, for all $f, g \in K$, as a mapping from $\mathbb{R}_+$ into Y.

DEFINITION 2.3. Let B be a subset of K. We say that the set-valued mapping $C : K \rightsquigarrow Y$ has a closed graph with respect to B if for every net $(f_i) \subset K$ and $(y_i) \subset Y$ such that $y_i \in C(f_i)$, (f_i) converge to $f \in B$ with respect to the topology of pointwise convergence (w.r.t.p.c.) and y_i converge to $y \in Y$, then $y \in C(f)$.

LEMMA 2.1. *Let $T : K \to X$ be C-pseudomonotone and hemicontinuous, and the graph of $Y \setminus C(\cdot)$ be closed with respect to $B \subset K$. Then the following two assumptions are equivalent:*

(i) *There exists* $f \in B$ *such that*

$$\langle g - f, T(f)\rangle \notin C(f), \forall g \in K.$$

(ii) *There exists* $f \in B$ *such that*

$$\langle g - f, T(g)\rangle \notin C(f), \forall g \in K.$$

Proof. We need to prove only $(ii) \Rightarrow (i)$ because the reverse implication follows from the C-pseudomonotonicity of T.

Let us suppose that (ii) holds. Then

$$\langle tg + (1-t)f - f, T(tg + (1-t)f)\rangle \notin C(f)$$

for all $g \in K$ and $t \in (0, 1)$. Hence

$$\langle g - f, T(f + t(g - f))\rangle \notin C(f),$$

which means that

$$\langle g - f, T(f + t(g - f))\rangle \in Y \setminus C(f).$$

The hemicontinuity of T and the closedness of the graph of $Y \setminus C(\cdot)$ with respect to B imply that

$$\langle g - f, T(f)\rangle \in Y \setminus C(f).$$

DEFINITION 2.4. Let B be a compact (w.r.t.p.c.) subset of K. The mapping $T : K \to X$ is said to be coercive with respect to B, if there exists $g_0 \in B$ such that

$$\langle g_0 - f, T(f)\rangle \in C(f), \forall f \in K \setminus B.$$

We can state now our result regarding the solvability of (OVI).

THEOREM 2.1. *Let the mapping* $T : K \to X$ *be* C*-pseudomonotone, hemicontinuous, coercive with respect to the compact set* $B \subset K$ *and let us suppose that the set-valued mapping* $Y \setminus C(\cdot)$ *has closed graph with respect to* $B \subset K$*. Then the variational inequality* (OVI) *is solvable.*

Proof. We define the set-valued mappings $T_1, T_2 : K \rightsquigarrow X$ by

$$T_1(g) = \{f \in K : \langle g - f, T(f)\rangle \notin C(f)\},$$
$$T_2(g) = \{f \in B : \langle g - f, T(g)\rangle \notin C(f)\}.$$

Using the fact that the sets $C(f)$ are convex and do not contain the origin we find that T_1 is a KKM mapping (similar to [4, 19]), which means that for all $g_1, \ldots, g_n \in K$ hold

$$\operatorname{conv}\{g_1, \ldots, g_n\} \subset \bigcup_{i=1}^{n} T_1(g_i).$$

Hence, the mapping $\overline{T}_1 : K \to X$, defined by $\overline{T}_1(f) = \overline{T_1(f)}$, the closure w.r.t.p.c., is also a KKM-mapping. The coercivity of T with respect to B implies that $\overline{T_1(g_0)} \subset B$ which means that $\overline{T_1(g_0)}$ is compact w.r.t.p.c. Using the Ky Fan lemma [6], we get

$$\bigcap_{g\in K} \overline{T_1(g)} \neq \emptyset.$$

We will prove that

$$\bigcap_{g\in K} \overline{T_1(g)} \subset T_2(h), \forall h \in K.$$

For, let us observe first that

$$\bigcap_{g\in K} \overline{T_1(g)} = \bigcap_{g\in K} \left(\overline{T_1(g)} \cap \overline{T_1(g_0)}\right) \subset \bigcap_{g\in K} \overline{T_1(g)} \cap B \subset B.$$

Let

$$f \in \bigcap_{g\in K} \overline{T_1(g)}.$$

Then $f \in B \cap \overline{T_1(g)}$ for all $g \in K$.

Let us choose $h \in K$ arbitrarily. Then there exists a net $(f_i^h) \subset T_1(h)$ such that f_i^h converge to $f \in B$ w.r.t.p.c.
The C-pseudomonotonicity of T implies that

$$\langle h - f_i^h, T(h)\rangle \in Y \setminus C(f_i^h)$$

and with the aid of the closedness of the graph of $Y \setminus C(\cdot)$ with respect to B we get

$$\langle h - f, T(h)\rangle \in Y \setminus C(f).$$

So, $f \in T_2(h)$ and since h was chosen arbitrarily, we get that $f \in T_2(h)$, for all $h \in K$. Hence

$$\emptyset \neq \bigcap_{g\in K} \overline{T_1(g)} \subset \bigcap_{g\in K} T_2(g) \subset B.$$

Using Lemma 3.1 we get

$$\bigcap_{g\in K} T_2(g) = \bigcap_{g\in K} T_1(g),$$

so

$$\bigcap_{g\in K} T_1(g) \neq \emptyset$$

which means that (VVI) has a solution.

We will prove now the solvability of (VVI) under weaker assumptions than in Theorem 1.1.

COROLLARY 2.1. *Let Z, W be real Banach spaces. Let K be a nonempty closed, convex subset of Z. Let $C_1 : K \rightsquigarrow W$ be such that, for each $z \in K$, $C_1(z) \neq W$ is a convex cone with int$C_1(z) \neq \emptyset$, and let $W \setminus (-\text{int}C_1(\cdot))$ having a weakly closed graph in $Z \times W$. Suppose that $T_1 : K \to (Z, W)^*$ is weakly C_1-pseudo-monotone, hemicontinuous (from $\mathbb{R}_+$ to the weak topology of W), and coercive. Then the (VVI) has a solution.*

Proof. Let us consider $X = (Z, W)^*$ as the Banach space of the linear and continuous mappings between two Banach spaces and $Y = (W, \sigma(W, W^*))$, i.e., W endowed with its weak topology. In this way we can consider W as a locally convex space.

We identify $z \in Z$ with $f_z \in ((Z, W)^*, W)^*$, defined by $f_z(l) = \langle l, z\rangle$ for all $l \in (Z, W)^*$.

We consider $K \subset Z \subset ((Z, W)^*, W)^*$ and the mapping $T : K \to (Z, W)^*$ defined by $T(f_z) = T_1(z)$.

The weak-compactness of B in Z imply the compactness of B w.r.t.p.c. in $(X, Y)^*$. Indeed, if $(f_{z_i}) \in B \subset ((Z, W)^*, W)^*$ is a net, then $(z_i) \in B \subset Z$ and we can choose a weakly converging subnet $z_j \rightharpoonup z \in B$. Hence $f_{z_j} \to f_z$ w.r.t.p.c.

We consider $C(f_z) = -\text{int}C_1(z)$ for all $z \in K$. The weak C_1-pseudomonotonicity of T on K implies the C-pseudomonotonicity of T on K.

Since the domain and the range of the mapping $t \mapsto \langle g - f, T(f + t(g - f))$ are $[0, 1]$ and Y, the hemicontinuity of T is fulfilled.

Let us prove that $Y \setminus C(\cdot)$ has a closed graph with respect to B. For this let $f_{z_i} \in K$ converging to $f_z \in B$ w.r.t.p.c. and $w_i \in C(f_{z_i})$ such that $w_i \rightharpoonup w$. Then

$$\langle l, z_i\rangle \rightharpoonup \langle l, z\rangle, \forall l \in (Z, W)^*.$$

So,

$$\langle w^* \circ l, z_i\rangle \to \langle w^* \circ l, z\rangle, \forall l \in (Z, W)^*, w^* \in W^*,$$

and hence

$$\langle z^*, z_i\rangle \to \langle z^*, z\rangle, \forall z^* \in Z^*,$$

which means that $z_i \rightharpoonup z$. Using the weak closedness of the graph of $W \setminus (-\text{int}C_1(\cdot))$ we get that $w \in W \setminus (-\text{int}C_1(z))$ and from here $w \in Y \setminus C(f_z)$.

So we can use Theorem 2.1 to find a solution for the problem (VVI).

3. The Goldman test and scalar variational inequalities

First we recall a theorem which is a generalization of the well-known Alaoglu-Bourbaki theorem regarding the weak-compactness of subsets of the dual of a Banach space.

THEOREM 3.1. (*Goldman* [8, 15]) *If X is a real topological vector space, Y is a real Hausdorff topological vector space, $U \subset X$ is a subset with nonempty interior, $M \subset Y$ is compact, then the set*

$$\mathcal{A} = \left\{ A \in (X, Y)^* : A(U) \subset M \right\}$$

is compact w.r.t.p.c.

This result can be used to verify the compactness of the set B in Theorem 3.2. Let us consider, for example, the case when $Y = \mathbb{R}$, X is a Hausdorff topological vector space, $K \subset X^*$ is nonempty and convex. Let $T : K \to X$ and we consider the following variational inequality:

$$\begin{cases} \text{Find } f_0 \in K \text{ such that} \\ \langle f - f_0, T(f_0) \rangle \geqslant 0, \forall f \in K. \end{cases} \tag{VI_1}$$

Theorem 2.1 reduces to the following result.

THEOREM 3.2. *Let X be a Hausdorff topological vector space, $K \subset X^*$ be nonempty and convex. Let B be convex, equicontinuous and closed w.r.t.p.c. Let $T : K \to X$ be pseudomonotone, hemicontinuous and coercive with respect to B. We suppose that T has closed graph with respect to B.*

Then the variational inequality (VI_1) has a solution.

Proof. We only need to show that B is compact with respect to the toplogy of pointwise convergence. Indeed, the equicontinuity shows that for $\varepsilon > 0$ and $M = [-\varepsilon, \varepsilon]$

$$U = \bigcap_{f \in B} f^{-1}([-\varepsilon, \varepsilon])$$

is a neighborhood of the origin in X. Then

$$B \subset \left\{ f \in X^* : f(U) \subset [\varepsilon, \varepsilon] \right\}.$$

Using Theorem 3.1, we realize that B as a closed subset of a compact set it is compact with respect ot the topology of pointwise convergence.

We will now give an example showing the utility of our proposal.

EXAMPLE 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions:

(i) $F(\cdot, r)$ is measurable for all $r \in \mathbb{R}$.
(ii) $F(\omega, \cdot)$ is continuous for a.e. $\omega \in \Omega$.
(iii) $F(\omega, \cdot)$ is monotone nondecreasing for all $\omega \in \Omega$.
(iv) $F(\cdot, r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$.

We consider the Nemitski-operator defined by F, namely

$$T(f)(\omega) = F(\omega, f(\omega))$$

which in this case maps $L^\infty(\Omega)$ into $L^1(\Omega)$. Moreover, T is continuous, bounded and monotone. Let $K \subset L^\infty(\Omega) = L^1(\Omega)^*$ be a closed, convex, bounded set and let us consider the following variational inequality:

$$\begin{cases} \text{Find } f_0 \in K \text{ such that} \\ \langle f - f_0, T(f_0)\rangle = \\ \int_\Omega F(\omega, f_0(\omega))(f(\omega) - f_0(\omega))\, d\omega \geqslant 0, \forall f \in K. \end{cases} \tag{VI_2}$$

K being weakly compact, we can use Theorem 3.2 to prove that, under assumptions $(i) - (iv)$, the problem (VI_2) has a solution.

4. B-pseudomonotone mappings

Let us consider the setting of the problem (OVI).

DEFINITION 4.1. The mapping $T : K \subset (X, Y)^* \to X$ is said to be B-pseudomonotone with respect to the set-valued mapping C if for each net $(f_i) \subset K$ and $f, g \in (X, Y)^*$ such that $f_i \to f$ w.r.t.p.c. and

$$\langle (1 - \lambda)f + \lambda g - f_i, T(f_i)\rangle \in Y \setminus C(f_i), \forall \lambda \in [0, 1], \forall f_i \tag{4.1}$$

we have

$$\langle g - f, T(f)\rangle \in Y \setminus C(f).$$

This B-pseudomonotonicity generalizes the pseudomonotonicity introduced by Brézis [1, 10, 18] in order to solve variational inequalities. Let us recall the definition of this notion in the context of the problem (VI) presented in the first section.

The mapping $T : K \to X^*$ is said to be pseudomonotone if for every net $(x_i)\ subset X$ and $x, y \in X$ such that $x_i \rightharpoonup x$ and $\liminf \langle T(x_i), x - x_i\rangle \geqslant 0$ we have

$$\limsup \langle T(x_i), y - x_i\rangle \leqslant \langle T(x), y - x\rangle.$$

From this definition we can deduce that if T is pseudomonotone, then

$$x_i \rightharpoonup x \text{and} \langle T(x_i), (1 - \lambda)x + \lambda y - x_i\rangle \geqslant 0, \forall \lambda \in [0, 1], \forall x_i$$

imply that $\langle T(x), y - x\rangle \geqslant 0$.

It is well known that a monotone and hemicontinuous mapping $T : K \to X^*$, where $K \subset X$ is closed and convex, is pseudomonotone in the sense of Brézis. The following proposition generalizes this result.

PROPOSITION 4.1. *Let us consider the setting of* (OVI) *and let us suppose that:*
(a) *T is C-pseudomonotone and hemicontinuous.*
(b) *The graph of $Y \setminus C(\cdot)$ is closed.*
Then T is B-pseudomonotne with respect to C.

Proof. Let us consider the set-valued mappings $T_1, T_2 : K \to (X, Y)^*$ defined by

$$T_1(g) = \{f \in K : \langle g - f, T(f)\rangle \in Y \setminus C(f)\}$$

and

$$T_2(g) = \{f \in K : \langle g - f, T(g)\rangle \in Y \setminus C(f)\}.$$

In order to prove the B-pseudomonotonicity of T, we have to prove that for each line segment D we have

$$\overline{\bigcap_{g \in K\cap D} T_1(g)} \bigcap D \subset \overline{\bigcap_{g \in K\cap D} T_2(g)} \bigcap D \subset$$

$$\subset \left(\bigcap_{g \in K\cap D} T_2(g)\right) \bigcap D = \left(\bigcap_{g \in K\cap D} T_1(g)\right) \bigcap D.$$

The first inclusion is due to the C-pseudomonotonicity of T, because $T_1(g) \subset T_2(g)$ for all $g \in K$.

The equality is implied by Lemma 2.1 in the case of $B = K = K \cap D$.

We have to prove only the second inclusion. For, let

$$f \in \overline{\bigcap_{g \in K\cap D} T_2(g)} \bigcap D$$

and $f_i \to f$ w.r.t.p.c. such that

$$f_i \in \bigcap_{g \in K\cap D} T_2(g).$$

Hence

$$\langle g - f_i, T(g)\rangle \in Y \setminus C(f_i), \forall g \in K \cap D.$$

Assumption (b) implies that

$$\langle g - f, T(g)\rangle \in Y \setminus C(f),$$

whence

$$f \in \bigcap_{g \in K\cap D} T_2(g) \bigcap D.$$

LEMMA 4.1. [2] *Let V be a Hausdorff topological vector space, $K \subset V$ and $T_1 : K \rightsquigarrow V$ such that:*

(i) *$\overline{T_1(v_0)}$ is compact for some $v_0 \in V$.*
(ii) *T_1 is a KKM-mapping.*
(iii) *For every $v \in K$, the intersection of $T_1(v)$ with any finite dimensional subspace is closed.*
(iv) *For every line segment D of V, we have*

$$\overline{\bigcap_{v \in K \cap D} T_1(v)} \bigcap D = \left(\bigcap_{v \in K \cap D} T_1(v) \right) \bigcap D.$$

Then

$$\bigcap_{v \in K} T_1(v) \neq \emptyset.$$

THEOREM 4.1. *Let us consider our setting for the p roblem (OVI) and let us suppose that:*
(1) The intersection of K with any finite dimensional subspace of $(X, Y)^*$ is closed w.r.t.p.c.
(2) The graph of $Y \setminus C(\cdot)$ is closed.
(3) T is continuous on the finite dimensional subspaces of $(X, Y)^*$.
(4) T is coercive with respect to a compact, convex $B \subset (X, Y)^*$.
(5) T is B-pseudomonotone with respect to the set-valued mapping C.
Then there exists $f_0 \in B$ such that

$$\langle g - f_0, T(f_0) \rangle \notin C(f_0), \forall g \in K.$$

Proof. Let $T_1 : K \rightsquigarrow X$ be defined by

$$T_1(f) = \{ f \in K : \langle g - f, T(f) \rangle \in Y \setminus C(f) \}.$$

In the same manner as we did in the proof of Theorem 2.1 we can prove that:
– $\overline{T_1(g_0)}$ is compact,
– T_1 is a KKM-mapping.
In order to prove that assumption (iii) of Lemma 4.1 is satisfied, let S be a finite dimensional subspace of $(X, Y)^*$ and let $g \in K$. Then

$$T_1 \bigcap S = \{ f \in K \cap S : \langle g - f, T(f) \rangle \in Y \setminus C(f) \}.$$

Let $(f_n) \subset T_1(g) \cap S$ such that $f_n \to f$. Since $K \cap S$ is closed, it follows that $f \in K \cap S$.

It is enough to consider the case when S is two dimensional. In this case there exists $(\lambda_n), (\mu_n) \subset \mathbb{R}$, $h, k \in (X, Y)^*$ such that $g - f_n = \lambda_n h + \mu_n k$ and $g - f =$

$\lambda h + \mu k$. We have $\lambda_n \to \lambda$, $\mu_n \to \mu$ and

$$\begin{aligned}&\langle g - f_n, T(f_n)\rangle - \langle g - f, T(f)\rangle \\ &\quad = \langle g - f_n, T(f_n) - T(f)\rangle + \langle f - f_n, T(f)\rangle \\ &\quad = \langle \lambda_n h + \mu_n k, T(f_n) - T(f)\rangle + \langle(\lambda_n - \lambda)h \\ &\qquad + (\mu_n - \mu)k, T(f)\rangle \to 0, n \to \infty.\end{aligned}$$

Assumption (2) implies now that $\langle g - f, T(f)\rangle \in Y \setminus C(f)$.

Assumption (iv) of Lemma 4.1 is implied by assumption (5) of Theorem 4.1. In order to prove this, it is enough to show that

$$f \in \overline{\bigcap_{\lambda\in[0,1]} T_1((1-\lambda)f + \lambda g)} \bigcap [f, g]$$

implies

$$f \in \bigcap_{\lambda\in[0,1]} T_1((1-\lambda)f + \lambda g) \bigcap [f, g].$$

Let (f_i) be a net such that

$$f_i \in \bigcap_{\lambda\in[0,1]} T_1((1-\lambda)f + \lambda g) \text{and} f_i \to f.$$

Hence

$$\langle(1-\lambda)f + \lambda g - f_i, T(f_i)\rangle \in Y \setminus C(f_i), \forall f_i,$$

therefore, by the B-pseudomonotonicity of T, we have

$$\langle g - f, T(f)\rangle \in Y \setminus C(f),$$

which means that

$$f \in \bigcap_{\lambda\in[0,1]} T_1((1-\lambda)f + \lambda g) \bigcap [f, g].$$

Assumptions $(i) - (iv)$ of Lemma 4.1 are satisfied by T_1, so

$$\bigcap_{g\in K} T_1(g) \neq \emptyset,$$

and hence there exists a solution of the problem (OVI).

REMARK 4.1. If T carries any finite dimensional subset of K into finite dimensional subsets, than it is enough to suppose that the intersection of the graph of $Y \setminus C(\cdot)$ with finite dimensional subspaces is closed. In this case the subsets $Y \setminus C(f)$ needn't be closed, which due to Remark 1.1 is an important property if Y is infinite dimensional.

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